NONDIFFERENTIABLE MINMAX FRACTIONAL PROGRAMMING WITH SQUARE ROOT TERMS

Anton BĂTĂTORESCU, Vasile PREDA and Miruna BELDIMAN
University of Bucharest, Faculty of Mathematics and Computer Science
14 Academiei str., 010014 - Bucharest, Romania

Abstract

We establish necessary and sufficient optimality condition for a class of nondifferentiable minmax fractional programming problems with square root terms involving (η, ρ, θ)-invex functions. Subsequently, we apply the optimality condition to formulate a parametric dual problem and we prove weak duality, strong duality, and strict converse duality theorems.

1 Introduction

Let us consider the following continuous differentiable mappings:

$$\begin{aligned} f &: \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \quad h : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}, \\ g &: \mathbb{R}^n \to \mathbb{R}^p, \end{aligned}$$

with $g = (g_1, \cdots, g_p)$. We denote

$$\mathcal{P} = \{x \in \mathbb{R}^n \mid g_j(x) \le 0,\ j = 1, 2, \cdots, p\} \tag{1}$$

and consider $Y \subseteq \mathbb{R}^m$ to be a compact subset of $\mathbb{R}^m$. Let B_r, $r = \overline{1, \beta}$, and D_q, $q = \overline{1, \delta}$, be $n \times n$ positive semidefinite matrices such that for each $(x, y) \in \mathcal{P} \times Y$, we have:

$$f(x, y) + \sum_{r=1}^{\beta} \sqrt{x^\top B_r x} \ge 0 \quad \text{and} \quad h(x, y) - \sum_{q=1}^{\delta} \sqrt{x^\top D_q x} > 0.$$

We consider the following minmax fractional programming problem:

$$\inf_{x \in \mathcal{P}} \sup_{y \in Y} \left(f(x, y) + \sum_{r=1}^{\beta} \sqrt{x^\top B_r x} \right) \left(h(x, y) - \sum_{q=1}^{\delta} \sqrt{x^\top D_q x} \right)^{-1} \tag{P}$$

For $\beta = \delta = 1$, this problem was studied by Lai et al. [4], and further, if $B_1 = D_1 = 0$, (P) is a differentiable minmax fractional programming problem which has been studied by Liu and Wu [5]. Many authors investigated the optimality conditions and duality theorems for minmax (fractional) programming problems. For details, one can consult [4, 7]. Problems which contain square root terms were first studied by Mond [6]. Some extensions of Mond's results were obtained, for example, by Chandra et al. [2], Zhang and Mond [12], Preda and Köller [8].

In an earlier work, under conditions of convexity, Schmittendorf [10] established necessary and sufficient optimality conditions for the problem:

$$\inf_{x \in \mathcal{P}} \sup_{y \in Y} \psi(x, y), \tag{P1}$$

where $\psi : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}$ is a continuous differentiable mapping.

2 Notations and Preliminary Results

Throughout this paper, we denote by $\mathbb{R}^n$ the n-dimensional Euclidean space and by $\mathbb{R}^n_+$ its non-negative orthant. Let us consider the set $\mathcal{P}$ defined by (1), and for each $x \in \mathcal{P}$, we define

$$\begin{aligned}
J(x) &= \{j \in \{1,2,\cdots,p\} \mid g_j(x) = 0\},\\
Y(x) &= \left\{ y \in Y \;\middle|\; \frac{f(x,y) + \sum_{r=1}^{\beta} \sqrt{x^\top B_r x}}{h(x,y) - \sum_{q=1}^{\delta} \sqrt{x^\top D_q x}} = \sup_{z \in Y} \frac{f(x,z) + \sum_{r=1}^{\beta} \sqrt{x^\top B_r x}}{h(x,z) - \sum_{q=1}^{\delta} \sqrt{x^\top D_q x}} \right\},\\
K(x) &= \left\{ (s,t,\bar{y}) \in \mathbb{N} \times \mathbb{R}^s_+ \times \mathbb{R}^{ms} \;\middle|\; \begin{array}{c} 1 \leq s \leq n+1,\ \sum_{i=1}^{s} t_i = 1, \\ \text{and } \bar{y} = (\bar{y}_1, \cdots, \bar{y}_s) \in \mathbb{R}^{ms} \\ \text{with } \bar{y}_i \in Y(x),\ i = \overline{1,s} \end{array} \right\}.
\end{aligned}$$

Since f and h are continuous differentiable functions and Y is a compact set in $\mathbb{R}^m$, it follows that for each $x_0 \in \mathcal{P}$, we have $Y(x_0) \neq \emptyset$, and for any $\bar{y}_i \in Y(x_0)$, we denote

$$k_0 = \left(f(x_0, \bar{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_0^\top B_r x_0} \right) \left(h(x_0, \bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_0^\top D_q x_0} \right)^{-1}. \tag{2}$$

Let A be an $m \times n$ matrix and let M, M_i, $i = 1, \cdots, k$, be $n \times n$ symmetric positive semidefinite matrices.

Lemma 1 [11] *We have*

$$Ax \geq 0 \;\Rightarrow\; c^\top x + \sum_{i=1}^{k} \sqrt{x^\top M_i x} \geq 0,$$

if and only if there exist $y \in \mathbb{R}^m_+$ *and* $v_i \in \mathbb{R}^n$, $i = \overline{1,k}$, *such that*

$$Av_i \geq 0, \quad v_i^\top M_i v_i \leq 1,\ i = \overline{1,k}, \quad A^\top y = c + \sum_{i=1}^{k} M_i v_i.$$

Lemma 2 [10] *Let* x_0 *be a solution of the minmax problem (P1) and the vectors* $\nabla g_j(x_0)$, $j \in J(x_0)$ *are linearly independent. Then there exist a positive integer* s, $1 \leq s \leq n+1$, *real numbers* $t_i \geq 0$, $i = \overline{1,s}$, $\mu_j \geq 0$, $j = \overline{1,p}$, *and vectors* $\bar{y}_i \in Y(x_0)$, $i = \overline{1,s}$, *such that*

$$\sum_{i=1}^{s} t_i \nabla_x \psi(x_0, \bar{y}_i) + \sum_{j=1}^{p} \mu_j \nabla g_j(x_0) = 0, \quad \mu_j g_j(x_0) = 0, \quad j = \overline{1,p}, \quad \sum_{i=1}^{s} t_i \neq 0.$$

Now we give the definitions of (η, ρ, θ)-quasi-invexity and (η, ρ, θ)-pseudo-invexity as extensions of the invexity notion.

Definition 3 *A differentiable function* $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ *is* (η, ρ, θ)*-invex at* $x_0 \in C$ *if there exist functions* $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ *and* $\rho \in \mathbb{R}$ *such that*

$$\varphi(x) - \varphi(x_0) \geq \eta(x, x_0)^\top \nabla \varphi(x_0) + \rho \theta(x, x_0).$$

If $-\varphi$ *is* (η, ρ, θ)*-invex at* $x_0 \in C$, *then* φ *is called* (η, ρ, θ)*-incave at* $x_0 \in C$.
If the inequality holds strictly, then φ *is called to be strictly* (η, ρ, θ)*-invex.*

Definition 4 *A differentiable function* $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ *is* (η, ρ, θ)*-pseudo-invex at* $x_0 \in C$ *if there exist functions* $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ *and* $\rho \in \mathbb{R}$ *such that the following hold:*

$$\eta(x, x_0)^\top \nabla \varphi(x_0) \geq -\rho \theta(x, x_0) \implies \varphi(x) \geq \varphi(x_0), \quad \forall x \in C,$$

Definition 5 *A differentiable function* $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ *is strictly* (η, ρ, θ)*-pseudo-invex at* $x_0 \in C$ *if there exist functions* $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ *and* $\rho \in \mathbb{R}$ *such that the following hold:*

$$\eta(x, x_0)^\top \nabla\varphi(x_0) \geq -\rho\theta(x, x_0) \implies \varphi(x) > \varphi(x_0), \quad \forall x \in C,\ x \neq x_0.$$

Definition 6 *A differentiable function* $\varphi : C \subseteq \mathbb{R}^n \to \mathbb{R}$ *is* (η, ρ, θ)*-quasi-invex at* $x_0 \in C$ *if there exist functions* $\eta : C \times C \to \mathbb{R}^n$, $\theta : C \times C \to \mathbb{R}_+$ *and* $\rho \in \mathbb{R}$ *such that the following hold:*

$$\varphi(x) \leq \varphi(x_0) \implies \eta(x, x_0)^\top \nabla\varphi(x_0) \leq -\rho\theta(x, x_0), \quad \forall x \in C.$$

3 Necessary and Sufficient Optimality Conditions

For any $x \in \mathcal{P}$, let us denote the following index sets:

$$\begin{aligned}
\mathcal{B}(x) &= \{r \in \{1, 2, \cdots, \beta\} \mid x^\top B_r x > 0\}, \\
\overline{\mathcal{B}}(x) &= \{1, 2, \cdots, \beta\} \setminus \mathcal{B}(x) = \{r \mid x^\top B_r x = 0\}, \\
\\
\mathcal{D}(x) &= \{q \in \{1, 2, \cdots, \delta\} \mid x^\top D_q x > 0\}, \\
\overline{\mathcal{D}}(x) &= \{1, 2, \cdots, \delta\} \setminus \mathcal{D}(x) = \{q \mid x^\top D_q x = 0\}.
\end{aligned}$$

Using Lemma 2, we may prove the following necessary optimality conditions for problem (P).

Theorem 7 (Necessary Condition) *If* x_0 *is an optimal solution of problem (P) for which* $\overline{\mathcal{B}}(x_0) = \emptyset$, $\overline{\mathcal{D}}(x_0) = \emptyset$, *and* $\nabla g_j(x_0)$, $j \in J(x_0)$ *are linearly independent, then there exist* $(s, \bar{t}, \bar{y}) \in K(x_0)$, $k_0 \in \mathbb{R}_+$, $w_r \in \mathbb{R}^n, r = \overline{1, \beta}$, $v_q \in \mathbb{R}^n, q = \overline{1, \delta}$, *and* $\bar{\mu} \in \mathbb{R}^p_+$ *such that*

$$\sum_{i=1}^{s} \bar{t}_i \left[\nabla f(x_0, \bar{y}_i) + \sum_{r=1}^{\beta} B_r w_r - k_0 \left(\nabla h(x_0, \bar{y}_i) - \sum_{q=1}^{\delta} D_q v_q\right)\right] + \sum_{j=1}^{p} \bar{\mu}_j \nabla g_j(x_0) = 0, \tag{3}$$

$$f(x_0, \bar{y}_i) + \sum_{r=1}^{\beta} \sqrt{x_0^\top B_r x_0} - k_0 \left(h(x_0, \bar{y}_i) - \sum_{q=1}^{\delta} \sqrt{x_0^\top D_q x_0}\right) = 0, \quad \forall\, i = \overline{1, s}, \tag{4}$$

$$\sum_{j=1}^{p} \bar{\mu}_j g_j(x_0) = 0, \tag{5}$$

$$\bar{t}_i \geq 0, \qquad \sum_{i=1}^{s} \bar{t}_i = 1, \tag{6}$$

$$\left.\begin{aligned}
w_r^\top B_r w_r \leq 1, \quad x_0^\top B_r w_r &= \sqrt{x_0^\top B_r x_0}, \quad r = \overline{1, \beta}, \\
v_q^\top D_q v_q \leq 1, \quad x_0^\top D_q v_q &= \sqrt{x_0^\top D_q x_0} \quad q = \overline{1, \delta}.
\end{aligned}\right\} \tag{7}$$

We notice that, in the above theorem, all matrices B_r and D_q are supposed to be positive definite. If at least one of $\overline{\mathcal{B}}(x_0)$ or $\overline{\mathcal{D}}(x_0)$ is not empty, then the functions involved in the objective function of problem (P) are not differentiable. In this case, the necessary optimality conditions still hold under some additional assumptions. For $x_0 \in \mathcal{P}$ and $(s, \bar{t}, \bar{y}) \in K(x_0)$ we define the following vector:

$$\alpha = \sum_{i=1}^{s} \bar{t}_i \left(\nabla f(x_0, \bar{y}_i) + \sum_{r \in \mathcal{B}(x_0)} \frac{B_r x_0}{\sqrt{x_0^\top B_r x_0}} - k_0 \left(\nabla h(x_0, \bar{y}_i) - \sum_{r \in \mathcal{D}(x_0)} \frac{D_q x_0}{\sqrt{x_0^\top D_q x_0}}\right)\right)$$

Now we define a set Z as follows:

$$Z_{\bar{y}}(x_0) = \left\{ z \in \mathbb{R}^n \;\middle|\; \begin{array}{l} z^\top \nabla g_j(x_0) \le 0, \; j \in J(x_0), \\ z^\top \alpha + \sum_{i=1}^{s} \bar{t}_i \left(\sum_{r \in \overline{\mathcal{B}}(x_0)} \sqrt{z^\top B_r z} + \sum_{q \in \overline{\mathcal{D}}(x_0)} \sqrt{z^\top \left((k_0)^2 D_q \right) z} \right) < 0 \end{array} \right\}$$

If one of the index sets involved in the above expressions is empty, then the correspondig sum vanishes.

Using Lemma 1, we establish the following result:

Theorem 8 *Let x_0 be an optimal solution of problem (P) and at least one of $\overline{\mathcal{B}}(x_0)$ or $\overline{\mathcal{D}}(x_0)$ is not empty. Let $(s, \bar{t}, \bar{y}) \in K(x_0)$ be such that $Z_{\bar{y}}(x_0) = \emptyset$. Then there exist vectors $w_r \in \mathbb{R}^n$, $r = \overline{1, \beta}$, $v_q \in \mathbb{R}^n$, $q = \overline{1, \delta}$, and $\bar{\mu} \in \mathbb{R}^p_+$ which satisfy the relations (3) - (7).*

For convenience, if a point $x_0 \in \mathcal{P}$ has the property that the vectors $\nabla g_j(x_0)$, $j \in J(x_0)$, are linear independent and the set $Z_{\bar{y}}(x_0) = \emptyset$, then we say that $x_0 \in \mathcal{P}$ satisfy a *constraint qualification.*

The results of Theorems 7 and 8 are the necessary conditions for the optimal solution of problem (P). Actually, the conditions (3) - (7) are also the sufficient optimality conditions for (P), for which we state the following result involving generalized invex functions, which are weaker assumptions than Lai et al. use in [4].

Theorem 9 (Sufficient Conditions) *Let $x_0 \in \mathcal{P}$ be a feasible solution of (P) and there exist a positive integer s, $1 \le s \le n+1$, $\bar{y}_i \in Y(x_0)$, $i = \overline{1, s}$, $k_0 \in \mathbb{R}_+$, defined by (2), $\bar{t} \in \mathbb{R}^s_+$, $w_r \in \mathbb{R}^n$, $r = \overline{1, \beta}$, $v_q \in \mathbb{R}^n$, $q = \overline{1, \delta}$, and $\bar{\mu} \in \mathbb{R}^p_+$ such that the relations (3) - (7) are satisfied. If any one of the following four conditions holds:*

(a) $f(\cdot, \bar{y}_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r$ *is (η, ρ_i, θ)-invex,* $h(\cdot, \bar{y}_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q$ *is (η, ρ_i', θ)-incave for $i = \overline{1, s}$,* $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is (η, ρ_0, θ)-invex, and* $\rho_0 + \sum_{i=1}^{s} \bar{t}_i (\rho_i + \rho_i' k_0) \ge 0$,

(b) $\overline{\Phi}(\cdot) = \sum_{i=1}^{s} \bar{t}_i \left[f(\cdot, \bar{y}_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r - k_0 \left(h(\cdot, \bar{y}_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q \right) \right]$ *is (η, ρ, θ)-invex and* $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is (η, ρ_0, θ)-invex, and $\rho + \rho_0 \ge 0$,*

(c) $\overline{\Phi}(\cdot)$ *is (η, ρ, θ)-pseudo-invex and* $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is (η, ρ_0, θ)-quasi-invex, and $\rho + \rho_0 \ge 0$,*

(d) $\overline{\Phi}(\cdot)$ *is (η, ρ, θ)-quasi-invex,* $\sum_{j=1}^{p} \bar{\mu}_j g_j(\cdot)$ *is strictly (η, ρ_0, θ)-pseudo-invex, $\rho + \rho_0 \ge 0$,*

then x_0 is an optimal solution of (P).

4 Duality

Let us consider the set $H(s, t, y)$ consisting of all $(z, \mu, k, v, w) \in \mathbb{R}^n \times \mathbb{R}^p_+ \times \mathbb{R}_+ \times \mathbb{R}^{n\delta} \times \mathbb{R}^{n\beta}$, where $v = (v_1, \cdots, v_\delta)$, $v_q \in \mathbb{R}^n$, $q = \overline{1, \delta}$, and $w = (w_1, \cdots, w_\beta)$, $w_r \in \mathbb{R}^n$, $r = \overline{1, \beta}$, which satisfy the following conditions:

$$\sum_{i=1}^{s} t_i \left[\nabla f(z, y_i) + \sum_{r=1}^{\beta} B_r w_r - k \left(\nabla h(z, y_i) - \sum_{q=1}^{\delta} D_q v_q \right) \right] + \sum_{j=1}^{p} \mu_j \nabla g_j(z) = 0, \tag{8}$$

$$\sum_{i=1}^{s} t_i \left[f(z,y_i) + \sum_{r=1}^{\beta} z^\top B_r w_r - k \left(h(z,y_i) - \sum_{q=1}^{\delta} z^\top D_q v_q \right) \right] \geq 0, \tag{9}$$

$$\sum_{j=1}^{p} \mu_j g_j(z) \geq 0, \tag{10}$$

$$(s,t,y) \in K(z) \tag{11}$$

$$w_r^\top B_r w_r \leq 1,\ r = \overline{1,\beta}, \quad v_q^\top D_q v_q \leq 1,\ q = \overline{1,\delta}. \tag{12}$$

The optimality conditions, stated in the preceding section for the minmax problem (P), suggest us to define the following dual problem:

$$\max_{(s,t,y)\in K(z)} \sup \{ k \mid (z,u,k,v,w) \in H(s,t,y) \} \tag{DP}$$

If, for a triplet $(s,t,y) \in K(z)$, the set $H(s,t,y) = \emptyset$, then we define the supremum over $H(s,t,y)$ to be $-\infty$. Further, we denote

$$\Phi(\cdot) = \sum_{i=1}^{s} t_i \left[f(\cdot,y_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r - k \left(h(\cdot,y_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q \right) \right]$$

Now, we can state the following weak duality theorem for (P) and (DP).

Theorem 10 (Weak Duality) *Let $x \in \mathcal{P}$ be a feasible solution of (P) and (x,μ,k,v,w,s,t,y) be a feasible solution of (DP). If any of the following four conditions holds:*

(a) $f(\cdot,y_i) + \sum_{r=1}^{\beta} (\cdot)^\top B_r w_r$ *is (η,ρ_i,θ)-invex,* $h(\cdot,y_i) - \sum_{q=1}^{\delta} (\cdot)^\top D_q v_q$ *is (η,ρ_i',θ)-incave for* $i = \overline{1,s}$, $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is (η,ρ_0,θ)-invex, and* $\rho_0 + \sum_{i=1}^{s} t_i(\rho_i + \rho_i' k) \geq 0$,

(b) $\Phi(\cdot)$ *is (η,ρ,θ)-invex and* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is (η,ρ_0,θ)-invex, and* $\rho + \rho_0 \geq 0$,

(c) $\Phi(\cdot)$ *is (η,ρ,θ)-pseudo-invex and* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is (η,ρ_0,θ)-quasi-invex, and* $\rho + \rho_0 \geq 0$,

(d) $\Phi(\cdot)$ *is (η,ρ,θ)-quasi-invex,* $\sum_{j=1}^{p} \mu_j g_j(\cdot)$ *is strictly (η,ρ_0,θ)-pseudo-invex,* $\rho + \rho_0 \geq 0$,

then

$$\sup_{y\in Y} \left(f(x,y) + \sum_{r=1}^{\beta} \sqrt{x^\top B_r x} \right) \left(h(x,y) - \sum_{q=1}^{\delta} \sqrt{x^\top D_q x} \right)^{-1} \geq k \tag{13}$$

Theorem 11 (Strong Duality) *Let x^* be an optimal solution of problem (P). Assume that x^* satisfies a constraint qualification for (P). Then there exist $(s^*,t^*,y^*) \in K(x^*)$ and $(x^*,\mu^*,k^*,v^*,w^*) \in H(s^*,t^*,y^*)$ such that $(x^*,\mu^*,k^*,v^*,w^*,s^*,t^*,y^*)$ is a feasible solution of (DP). If the hypotheses of Theorem 10 are also satisfied, then $(x^*,\mu^*,k^*,v^*,w^*,s^*,t^*,y^*)$ is an optimal solution for (DP), and both problems (P) and (DP) have the same optimal values.*

Theorem 12 (Strict Converse Duality) *Let x^* and $(\bar{z},\bar{\mu},\bar{k},\bar{v},\bar{w},\bar{s},\bar{t},\bar{y})$ be the optimal solutions of (P) and (DP), respectively, and that the hypotheses of Theorem 11 are fulfilled. If any one of the following three conditions holds:*

(a) *one of* $f(\cdot,\bar{y}_i)+\sum_{r=1}^{\beta}(\cdot)^\top B_r\bar{w}_r$ *is strictly* (η,ρ_i,θ)*-invex,* $h(\cdot,\bar{y}_i)-\sum_{q=1}^{\delta}(\cdot)^\top D_q\bar{v}_q$ *is strictly* (η,ρ_i',θ)*-incave for* $i=\overline{1,s}$*, or* $\sum_{j=1}^{p}\bar{\mu}_j g_j(\cdot)$ *is strictly* (η,ρ_0,θ)*-invex, and* $\rho_0+\sum_{i=1}^{s}\bar{t}_i\left(\rho_i+\rho_i'\bar{k}\right)\geq 0$;

(b) *either* $\sum_{i=1}^{s}\bar{t}_i\left[f(\cdot,\bar{y}_i)+\sum_{r=1}^{\beta}(\cdot)^\top B_r\bar{w}_r-\bar{k}\left(h(\cdot,\bar{y}_i)-\sum_{q=1}^{\delta}(\cdot)^\top D_q\bar{v}_q\right)\right]$ *is strictly* (η,ρ,θ)*-invex or* $\sum_{j=1}^{p}\bar{\mu}_j g_j(\cdot)$ *is strictly* (η,ρ_0,θ)*-invex, and* $\rho+\rho_0\geq 0$;

(c) *the function* $\sum_{i=1}^{s}\bar{t}_i\left[f(\cdot,\bar{y}_i)+\sum_{r=1}^{\beta}(\cdot)^\top B_r\bar{w}_r-\bar{k}\left(h(\cdot,\bar{y}_i)-\sum_{q=1}^{\delta}(\cdot)^\top D_q\bar{v}_q\right)\right]$ *is strictly* (η,ρ,θ)*-pseudo-invex and* $\sum_{j=1}^{p}\bar{\mu}_j g_j(\cdot)$ *is* (η,ρ_0,θ)*-quasi-invex, and* $\rho+\rho_0\geq 0$;

then $x^*=\bar{z}$*, that is,* $\bar{z}$ *is an optimal solution for problem (P) and*

$$\sup_{y\in Y}\left(f(\bar{z},y)+\sum_{r=1}^{\beta}\sqrt{\bar{z}^\top B_r\bar{z}}\right)\left(h(\bar{z},y)-\sum_{q=1}^{\delta}\sqrt{\bar{z}^\top D_q\bar{z}}\right)^{-1}=\bar{k}.$$

5 Special Cases

If we consider special cases of the results presented in this paper, we may retrieve some previous results obtained by other authors.

1. If we consider $\beta=\delta=1$, we obtain the results obtained by Lai et al. [4].
2. If $B_r=0$, $r=\overline{1,\beta}$, and $D_q=0$, $q=\overline{1,\delta}$, we obtain the results of Liu and Wu [5].
3. If the set Y is a singleton, $\beta=1$, $h\equiv 1$ and $D_q=0$, $q=\overline{1,\delta}$, we obtain the results presented respectively in Mond [6], Chandra et al. [2], Zhang and Mond [12], Preda and Köller [8].
4. For the case of the generalized fractional programming [1, 3], the set Y can be taken as the simplex $Y=\left\{y\in\mathbb{R}^m \;\middle|\; y_i\geq 0, \sum_{i=1}^{m}y_i=1\right\}$, $B_r=0$, $r=\overline{1,\beta}$, and $D_q=0$, $q=\overline{1,\delta}$, and

$$\frac{f(x,y)}{h(x,y)}=\left(\sum_{i=1}^{m}y_i f_i(x)\right)\left(\sum_{i=1}^{m}y_i h_i(x)\right)^{-1}.$$

In this case the dual (DP) reduces to the dual problem of [1].

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